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On the repulsion of an interface above a correlated substrate

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Abstract. We analyze a model of an interface fluctuating above a rough substrate. It is based on harmonic crystals, or lattice free fields, indexed by $\mathbb{Z}^d$, $d \geq 3$. The phenomenon for which we want to get precise quantitative estimates is the repulsion effect of the substrate on the interface: the substrate is itself a random field, but its randomness is *quenched* (this generalizes the widely considered case of a flat deterministic substrate). With respect to [2] in which the substrate has been taken to be an IID field, here the substrate is an harmonic crystal, as the interface, and as such it is strongly correlated. We obtain the leading asymptotic behavior of the model in the limit of a very extended substrate: we show in particular that, to leading order, the effect of an IID substrate cannot be distinguished from the effect of an harmonic crystal substrate. We observe however that, unlike in the IID substrate case, annealed and quenched models display sharply different features.

Keywords: harmonic crystals, entropic repulsion, Rough substrates, Quenched and Annealed models.

Mathematical subject classification: 82B24, 60K35, 60G15.

1 Introduction

1.1 Hard (flat) walls and interfaces

Let $\varphi = \{\varphi_x\}_{x\in\mathbb{Z}^d}$, $d \geq 3$, be a centered Gaussian field with

$$\mathbb{E}\left[\varphi_x\varphi_y\right] = G(x, y), \tag{1.1}$$

where $G(x, y) = \sum_{n=0}^{\infty} \mathbf{P}(X_n = y|X_0 = x)$, $X = \{X_n\}_{n=0,1,\dots}$ is a simple nearest neighbors random walk on $\mathbb{Z}^d$, so $G(\cdot, \cdot)$ is the Green function of the walk.

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We will refer to φ as *harmonic crystal*. A d–dimensional harmonic crystal may be viewed as a very simple model of a microscopic interface in a $(d+1)$–dimensional space. One of course is interested also in the cases of $d = 1$ and $d = 2$, but in these cases the walk is recurrent and the Green function exists only if boundary conditions are imposed: this is not just a mathematical technicality, but it is connected to important modelization aspects and the observed phenomena are often quantitatively and qualitatively different. In short, we look at *rigid* interfaces, as opposed to the *rough* harmonic crystal interfaces in lower dimension: we refer to [1], [7] and [12] for several details and further literature on these aspects. Here we mention the fact that much interest has been put into understanding quantitatively the effect of a hard wall, or of a more general forbidden region, on an interface. What one observes is a surprisingly strong repulsion effect, which goes under the name of entropic repulsion. Mathematically entropic repulsion poses very challenging and, to a certain extent, atypical problems connected to classical topics like Large Deviations and the analysis of extrema of random fields.

Let us be concrete and briefly review the precise results proven in [6]: chosen a bounded connected open set $D \subset \mathbb{R}^d$ containing the origin and such that ∂D is piecewise smooth, we consider the event

$$\Omega_N^+ = \{\varphi : \varphi_x \geq 0 \text{ for every } x \in D_N\}\,, \tag{1.2}$$

where $D_N = ND \cap \mathbb{Z}^d$. We are of course committing abuse of notation by not distinguishing between random variables and numerical variables when referring to φ (and later on when referring to σ). The capacity of D plays for us an important role and we recall its definition(s):

$$\begin{aligned}\mathrm{Cap}(D) &= \inf\left\{\frac{1}{2d}\|\partial f\|_2^2 \,:\, f \in C_0^\infty(\mathbb{R}^d;[0,\infty)),\ f(r) = 1 \text{ for all } r \in D\right\}\\ &= \sup_{f\in L^\infty(D)} \frac{\left(\int_D f(r)\,\mathrm{d}r\right)^2}{\iint_{D^2} f(r)f(r')R_d|r-r'|^{2-d}\,\mathrm{d}r\,\mathrm{d}r'},\end{aligned} \tag{1.3}$$

in which ∂ denotes the gradient in $\mathbb{R}^d$, $\|\cdot\|_2$ is the L^2–norm of $\cdot$ and

$$R_d := \lim_{x\to\infty} |x|^{d-2} G(0,x) \in (0,\infty). \tag{1.4}$$

See e.g. [4] for a proof of the equivalence of the two formulas for the capacity in (1.3). A proof of the existence of the limit in (1.4) can be found for example in [13]. The following result is proven in [6]:

Theorem 1.1. *Set* $G = G(0,0)$.

(1) *the Laplace asymptotics of the probability of* Ω_N^+ *is known:*

$$\lim_{N\to\infty} \frac{1}{N^{d-2}\log N} \log \mathbb{P}\left(\Omega_N^+\right) = -2G\mathrm{Cap}(D) \tag{1.5}$$

(2) *For any* $a < \sqrt{4G} < b$ *and any* $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}\left(\sum_{x\in D_N} \mathbb{1}_{(a,b)}\left(\varphi_x/\sqrt{\log N}\right) \le (1-\varepsilon)|D_N| \middle| \Omega_N^+ \right) = 0. \tag{1.6}$$

A few comments:

- It should be clear that $\mathbb{P}(\,\cdot\,|\Omega_N^+)$ yields a model for an interface above a flat hard wall.
- The first result, formula (1.5), should attract attention in particular for its anomalous normalization. The N^{d-2} term is directly connected to the slow decay of the correlations and it is already present for example in [4]. The $\log N$ correction is more mysterious and, in a sense, it catches the essence of the repulsion phenomenon, see the next point.
- The second result, formula (1.6), is telling us that the interface is repelled to infinity with a speed proportional to $\sqrt{\log N}$. It is this anomalously large repulsion that is responsible for the logarithmic correction in the speed of the Laplace asymptotics in (1.5).
- Catching the precise asymptotics has required a sharp understanding of the repulsion mechanism and this is interesting in itself.
- An alternative approach to harmonic crystals is the Gibbsian one, see [11, Ch. 13] and [12]. In particular the field φ that we defined is the only centered extremal Gibbs measure with respect to the (formal) Hamiltonian

$$H(\varphi) = \frac{1}{8d} \sum_{x,y:|x-y|=1} \left(\varphi_x - \varphi_y\right)^2. \tag{1.7}$$

  This field may be obtained from finite volume Gibbs measures with zero boundary conditions by taking infinite volume limits. Notice that the model is extremely sensitive to the boundary conditions: for instance setting the boundary conditions equal to a non zero constant one obtains a different limit measure. This is a consequence of the *continuum symmetry* enjoyed by (1.7), that can be restated by saying that the Hamiltonian depends *on the gradient of* φ and not on φ itself. This symmetry plays of course a central role in the entropic repulsion phenomenon. We refer to [8] for an analog of Theorem 1.1 in presence of boundary conditions.

1.2 Rough substrate: the IID case

Of course there is no reason to limit the attention to perfectly flat walls. In a recent work [2] we concentrated our attention on the case in which the wall is flat only on average, but local fluctuations are allowed. More precisely the level below which φ cannot go is site dependent and it is given by a typical realization of an IID centered field. What we show is that, even if the variance of the IID variables is arbitrarily small and in spite of the fact that the wall is *essentially flat*, the repulsion asymptotics may change, in the sense that the right–hand side of formulas (1.5) and (1.6) change, according to the tail asymptotics of the IID variables. Let us be more precise: let $\sigma = \{\sigma_x\}_{x\in\mathbb{Z}^d}$ be the IID field modeling the wall: σ and φ are independent. Here for simplicity we assume $\sigma_0 \sim \mathcal{N}(0, Q)$, $Q > 0$, but Theorem 1.2 below holds with no change for more general variables with *almost Gaussian upper tails*, see [2, Hypotheses H–2 and H–3]. The case in which the upper tail of σ_0 (by upper tail we mean the asymptotics of the probability that σ_0 is *positive* and large) is sub– or super–Gaussian is easier and less interesting for reasons that are probably intuitive, see [2].

Given $\sigma \in \mathbb{R}^{\mathbb{Z}^d}$ and $A \subset \mathbb{Z}^d$, the σ*–entropic repulsion event* on A is defined by

$$\Omega^+_{A,\sigma} = \{\varphi : \varphi_x \geq \sigma_x \text{ for every } x \in A\} . \tag{1.8}$$

We use the shortcut notation $\Omega^+_{N,\sigma} = \Omega^+_{D_N,\sigma}$.

The way in which we introduced the model up to now is clearly aiming at *quenched results*: we are choosing a $\mathbf{P}$–typical configuration σ and we keep it fixed (while φ is considered random). There is however another natural viewpoint: we may average both σ and φ and that is what we mean by *annealed* case. In this case, with abuse of notation, $\Omega^+_{A,\sigma}$ is rather the event $\{(\sigma, \varphi) : \varphi_x \geq \sigma_x \text{ for every } x \in A\}$.

Theorem 1.2. [2, Theorem 1.1] *We have that*

$$\lim_{N\to\infty} \frac{1}{N^{d-2}\log N} \log \mathbb{P}\left(\Omega^+_{N,\sigma}\right) = \lim_{N\to\infty} \frac{1}{N^{d-2}\log N} \log \mathbf{P}\otimes\mathbb{P}\left(\Omega^+_{N,\sigma}\right) \\ = -2(G+Q)\mathrm{Cap}(D). \tag{1.9}$$

$\mathbf{P}(\,\mathrm{d}\sigma)$–*a.s.*.

The point that we want to stress here is that quenched and annealed probability asymptotics coincide to leading order. This fact is exploited in the proof in [2], in the sense that there we have proven an annealed upper bound and a quenched lower bound, and the two coincide: a priori we know that *an upper bound on the probability asymptotics for an annealed model is also an upper bound on the probability asymptotics for the associated quenched model* (see Lemma 2.2 below) and, analogously, a quenched lower bound is also an annealed lower bound, and therefore the proof is completed with no further effort. We refer to these bounds as *quenched–annealed comparisons.*

Of course also a result analogous to Theorem 1.1 (part (2)), is available [2, Theorem 1.2]: for the measure $\mathbb{P}\left(\,\cdot\,\middle|\Omega^+_{N,\sigma}\right)$ one has to choose $a < \sqrt{4(G+Q)} < b$. It is not difficult to extract from this (or to prove directly) an annealed result saying that the φ field is repelled to height $\sqrt{4(G+Q)\log N}$ and that the σ field is left essentially unchanged.

1.3 Rough substrate: a strongly correlated case

It has been pointed out in [2] that the proof of Theorem 1.2, while possibly generalizable to weakly interacting σ–fields (with approximately Gaussian upper tails), breaks down if the σ field is strongly correlated. With the result that we present in this note we make that observation precise by dealing at the same time with a case of interest. Set up and notations coincide with the ones of § 1.2, with the (crucial) exception that this time σ is itself an harmonic crystal, in the sense that it has the same law as $s\varphi$, for $s \in \mathbb{R}$.

Theorem 1.3. *For the quenched model we have that for every s*

$$\lim_{N\to\infty} \frac{1}{N^{d-2}\log N} \log \mathbb{P}\left(\Omega^+_{N,\sigma}\right) = -2(G(1+s^2))\mathrm{Cap}(D), \tag{1.10}$$

$\mathbf{P}(\,\mathrm{d}\sigma) -$ *a.s., while for the annealed one, still for every s, we have that*

$$\lim_{N\to\infty} \frac{1}{N^{d-2}\log N} \log \mathbf{P}\otimes\mathbb{P}\left(\Omega^+_{N,\sigma}\right) = -2G\mathrm{Cap}(D). \tag{1.11}$$

The result that requires a proof is the first, that is (1.10), since the second one is an immediate consequence of Theorem 1.1, part (1): in fact $\varphi - \sigma \sim \sqrt{1+s^2}\varphi$ (here of course $\sim$ denotes the equivalence in law) and Ω_N^+ is invariant under the application $\varphi \longrightarrow c\varphi$, c a positive constant.

For what concerns instead (1.10) we separately prove lower and upper bounds. In view of the statement we want to prove, it should be clear that the quenched–annealed comparison strategy of [2] is doomed for $s \neq 0$. In spite of this, the proof that we present here is still based on a quenched–annealed comparison, but it is applied at a different stage of the proof (we do not try to explain this point further here, but invite the reader to have a look at Remark 2.3, where the rather intuitive procedure is summed up). On the other side the quenched lower bound is proven here by directly upgrading the proof in [2]: it is a matter of having a (sharp, to leading order) bound on the number of sites in which typical σ configurations exceed a level $\sqrt{\alpha \log N}$, $\alpha > 0$, in the domain D_N. It turns out that independent fields and harmonic crystals cannot be distinguished by looking, to leading order, at these quantities. This fact, due to the strong correlations of the harmonic crystal, may be somewhat surprising, but ultimately it lies behind several proofs on and around entropic repulsion problems in high dimension [12]. The situation changes, to a certain extent, if $d = 2$ [5].

In this note we also prove the following result on the path properties:

Proposition 1.4. *Fix $s \in \mathbb{R}$. For any $a < \sqrt{4(1+s^2)G} < b$ and any $\varepsilon > 0$*

$$\lim_{N\to\infty} \mathbb{P}\left(\sum_{x\in D_N} \mathbb{1}_{(a,b)}\left(\varphi_x/\sqrt{\log N}\right) \leq (1-\varepsilon)|D_N| \,\middle|\, \Omega_{N,\sigma}^+ \right) = 0, \qquad (1.12)$$

$\mathbf{P}(\,\mathrm{d}\sigma)$*–a.s..*

From this result we can see that also at the level of path properties quenched$\neq$ annealed. We give here a quick argument in the case $s = 1$: in the annealed case it is immediate to apply Theorem 1.1 (part 2) to see that the two fields are repelled approximately at a distance $\sqrt{8G \log N}$, like in the quenched case. However, by symmetry, the quenched path behavior (σ close to level zero and φ close to level $\sqrt{8G \log N}$ with probability close to one) is clearly incompatible with an annealed path behavior.

Remark 1.5. One can generalize the quenched results (1.10) and Proposition 1.4 to the case of φ and σ harmonic crystals defined via more general symmetric random walks and even to the interesting case in which two different random walks define respectively φ and σ. The precise framework that we have in mind is that of [4] and [3]. It suffices to replace $(1+s^2)G$ in both statements by $G+\tilde{G}$, with $\tilde{G} = \mathrm{var}\,(\sigma_0)$, while the capacity is still the one associated to the generator (or Laplacian) of the random walk defining φ. The proof of the probability upper bound in this case is slightly more involved, since $\varphi-\sigma$, under $\mathbf{P}\otimes\mathbb{P}$, is no longer an harmonic crystal. This difficulty is however overcome by following corresponding steps in the upper bound argument in [3]. For the annealed results in this set–up we have only partial results up to now and the problem seems to demand a new strategy.

2 The proofs

For sake of compactness we limit ourselves to $s = 1$. Only trivial changes are needed to handle the general case. The organization of this section is straightforward: in §2.1 and §2.2 we prove (1.10) and in §2.3 we prove Proposition 1.4.

2.1 Proof of (1.10): lower bound

With respect to earlier work, establishing that

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}\log N} \log \mathbb{P}\left(\Omega^+_{N,\sigma}\right) \geq -4G\mathrm{Cap}(D), \quad \mathbf{P}(\,\mathrm{d}\sigma) - \text{a.s.} \tag{2.1}$$

requires getting a (sharp, to leading order) upper bound on the number of high excursion of the σ field in the box D_N. This is provided by the following lemma and the rest of the proof is identical to the one in [2, §2].

Lemma 2.1. *For every $\alpha \in (0, 2dG)$ and every $\beta \in (0, \alpha)$, with*

$$A_{N,\alpha} := \left\{ \left| \left\{ x \in D_N : \sigma_x \geq \sqrt{\alpha \log N} \right\} \right| \geq |D_N|\, N^{-\frac{\beta}{2G}} \right\}, \tag{2.2}$$

we have that $\mathbf{P}(A_{N,\alpha}$ i.o.$) = 0$.

Proof. We write $\sigma = \sigma^{(1)} + \sigma^{(2)}$, with $\sigma^{(1)}$ and $\sigma^{(2)}$ independent Gaussian centered fields specified by

$$\begin{aligned} \mathbf{E}\left(\sigma^{(1)}_x \sigma^{(1)}_y\right) &= G_\varepsilon(x,y) \quad \text{and} \\ \mathbf{E}\left(\sigma^{(2)}_x \sigma^{(2)}_y\right) &= G(x,y) - G_\varepsilon(x,y), \quad x, y \in \mathbb{Z}^d, \end{aligned} \tag{2.3}$$

with $\varepsilon \in (0, 1)$ and $G_\varepsilon(\cdot, \cdot)$ is the Green function of the random walk $X^{(\varepsilon)} = \{X_n^{(\varepsilon)}\}_n$ that at each time has a probability ε of death (i.e. of jumping to an absorbing state that we add to the state space $\mathbb{Z}^d$) and with probability $(1-\varepsilon)/2d$ jumps to one of the nearest neighbors. It is easy to verify, for example via Fourier transform, that

- there exists $m(\varepsilon)$ such that $(0 <)G_\varepsilon(x, y) \leq \exp(-m(\varepsilon)|x - y|)$, at least if $|x - y|$ is sufficiently large;
- $G(\cdot, \cdot) - G_\varepsilon(\cdot, \cdot)$ is positive definite, in the sense of [11, Ch. 13], so it is a covariance (infinite dimensional) matrix. Moreover

$$G(0, 0) - G_\varepsilon(0, 0) \stackrel{\varepsilon \to 0}{\longrightarrow} 0.$$

We observe now that

$$\begin{aligned} \mathbf{P}\left(\max_{x \in D_N} \sigma_x^{(2)} \geq \sqrt{\delta N}\right) &\leq |D_N|\, \mathbf{P}\left(\sigma_0^{(2)} \geq \sqrt{\delta \log N}\right) \\ &\leq c N^{d-(\delta/2(G-G_\varepsilon))}, \end{aligned} \tag{2.4}$$

with $G_\varepsilon = G_\varepsilon(0, 0)$ and therefore, by Borel–Cantelli lemma, whenever $\delta > 2(d+2)(G - G_\varepsilon)$, $\mathbf{P}(\,\mathrm{d}\sigma)$–a.s. for N sufficiently large $\max_{x \in D_N} \sigma_x^{(2)}$ is smaller than $\sqrt{\delta \log N}$. In view of this we are left with proving Lemma 2.1 with σ replaced by $\sigma^{(1)}$ and G replaced by G_ε.

Unlike $G(\cdot, \cdot)$, $G_\varepsilon(\cdot, \cdot)$ is fast decaying, and so for every $\varepsilon > 0$ the sum $\Sigma_\varepsilon := \sum_{x,y} G_\varepsilon(x, y)$ is finite. This implies that the field $\sigma^{(1)}$ is hypercontractive [10] and therefore (see [6]) there exists $K = K(\Sigma_\varepsilon) \geq 1$ such that for every bounded measurable f

$$\mathbf{E}\left[\prod_{x \in D_N} f\left(\sigma_x^{(1)}\right)\right] \leq \left(\mathbf{E}\left[\left|f\left(\sigma_0^{(1)}\right)\right|^K\right]\right)^{\frac{|D_N|}{K}}. \tag{2.5}$$

Of course this property is the substitute for the independence.

Define $\widetilde{N}_\alpha = \left|\left\{x \in D_N : \sigma_x^{(1)} \geq \sqrt{\alpha \log N}\right\}\right|$ and choose

$$f(\,\cdot\,) = \exp\left(\mathbb{1}_{\{\,\cdot\, \geq \sqrt{\alpha \log N}\}}\right)$$

in (2.5), so that

$$\begin{aligned}
\mathbf{E}\left(\exp\left(\widetilde{N}_\alpha\right)\right) &= \mathbf{E}\left(\prod_{x\in D_N} \exp\left(\mathbb{1}_{\{\sigma_x^{(1)}\geq\sqrt{\alpha\log N}\}}\right)\right) \\
&\leq \left(\mathbf{E}\left(\exp\left(K\,\mathbb{1}_{\{\sigma_0^{(1)}>\sqrt{\alpha\log N}\}}\right)\right)\right)^{\frac{cN^d}{K}} \qquad (2.6)\\
&= \left(1+\left(e^K-1\right)\mathbf{P}\left(\sigma_0^{(1)}\geq\sqrt{\alpha\log N}\right)\right)^{cN^d/K} \\
&\leq \exp\left(c' N^{d-\frac{\alpha}{2G_\varepsilon}}\right),
\end{aligned}$$

where $c>0$ is so that $cN^d \leq |D_N|$ and $c'=c'(K)>0$. By Markov inequality

$$\mathbf{P}\left(\widetilde{N}_\alpha > N^{d-\frac{\beta}{2G_\varepsilon}}\right) \leq \exp\left(c'N^{d-\frac{\alpha}{2G_\varepsilon}} - N^{d-\frac{\beta}{2G_\varepsilon}}\right), \qquad (2.7)$$

and, since $\beta<\alpha<2dG_\varepsilon$, from the Borel–Cantelli lemma we deduce the assertion. □

2.2 Proof of (1.10): upper bound

The aim is to show that $\mathbf{P}(\,\mathrm{d}\sigma)$–a.s.

$$\limsup_{N\to\infty} \frac{1}{N^{d-2}\log N}\log\mathbb{P}\left(\Omega_{N,\sigma}^+\right) \leq -4G\mathrm{Cap}(D). \qquad (2.8)$$

The proof of (2.8) is based on a 2–scale dilution procedure and for this we need some notation, summed up in Figure 1. Choose an even natural number L and for $y\in 2L\mathbb{Z}^d$ set

$$B(y) = B_L(y) = \left\{x:\max_{i=1,\dots,d}|x_i-y_i| = L/2\right\}, \qquad (2.9)$$

($B(y)$ are the *walls* of the L–box surrounding y). We denote by Λ_c the set of $y\in 2L\mathbb{Z}^d$ such that $B(y)\subset D_N$ and we let $\Lambda=\bigcup_{y\in\Lambda_c} B(y)$. Clearly

$$\mathbb{P}\left(\Omega_{N,\sigma}^+\right) \leq \mathbb{P}\left(\Omega_{\Lambda\cup\Lambda_c,\sigma}^+\right) \leq \mathbb{P}\left(\Omega_{\Lambda,\sigma}^+\right). \qquad (2.10)$$

Let $q(z)=q_L(z)$ be the probability that a simple random walk leaving y hits $B(y)$ at z and for $\psi\in\mathbb{R}^{\mathbb{Z}^d}$ set $M_y(\psi)=\sum_{z\in B(y)} q(z)\psi_z$ ($M_y(\psi)$ is a weighted average of ψ on $B(y)$).

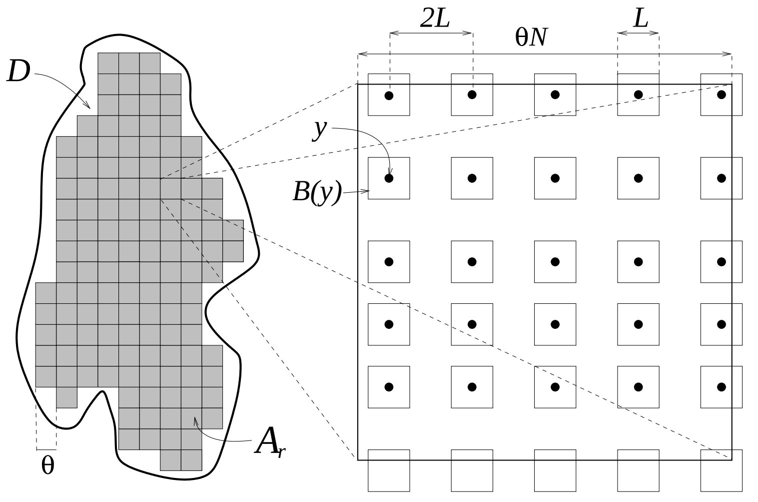

Figure 1: The 2–scale decomposition for the upper bound: on the left the macroscopic domain and the small ($\theta \ll 1$) macroscopic boxes decomposition; on the right we zoom $\times N$ on a small macroscopic box and the large ($L \gg 1$) microscopic boxes become visible.

As a second step we take $\theta > 0$ and we consider the *inner* θ*–discretization* of D: that is for $r \in \theta\mathbb{Z}^d$, set $A_r = r + [0, \theta)^d$ and define $I = I(\theta) = \{r \in \theta\mathbb{Z}^d : A_r \subset D\}$ (assume $I \neq \emptyset$). Now define $C_r = NA_r \cap \Lambda_c$ (the set of the black dots in Figure 1) and remark that $|C_r| = (N\theta/2L)^d(1 + o(1))$.

For every σ and for positive η and α let us consider the event

$$E^{\sigma}_{\eta,\alpha} = \Big\{\varphi : \text{there exists } r \in I \text{ such that} \\ \Big|\{y \in C_r : \varphi_y \le \sqrt{\alpha \log N} + \sigma_y\}\Big| \ge \eta|C_r|\Big\}. \tag{2.11}$$

The key point of the proof is to show that if $\alpha < 4G$ then the probability of $\Omega^+_{N,\sigma} \cap E^{\sigma}_{\eta,\alpha}$ is negligible, in the sense that for L sufficiently large it is smaller than $\exp(-cN^{d-2}\log N)$ for every c. We do not prove this directly, rather we make a detour via an annealed estimate that will serve our purposes since we have the following result (for a proof see for example [2, Corollary 3.2]):

Lemma 2.2. *Suppose that* $\{F_{N,\sigma}\}_N$ *is for every* σ *a sequence of events which are measurable with respect to* φ *and that* $\mathbb{1}_{F_{N,\sigma}}(\varphi)$ *is a measurable function jointly with respect to* φ *and* σ*. Suppose moreover that* $\mathbf{P} \otimes \mathbb{P}\left(F_{N,\sigma}\right) \le \exp(-a_N)$ *for* *a sequence* $\{a_N\}_N$ *of positive numbers such that* $\sum_N \exp(-\varepsilon a_N) < \infty$ *for every* $\varepsilon > 0$*, then* $\mathbf{P}(\,\mathrm{d}\sigma)$*–a.s.*

$$\limsup_{N\to\infty} \frac{1}{a_N} \log \mathbb{P}\left(F_{N,\sigma}\right) \le -1 \tag{2.12}$$

Aiming at bounding $\mathbb{P}\left(E^\sigma_{\eta,\alpha}\right)$, we estimate the $\mathbf{P}\otimes\mathbb{P}$–probability of the event

$$E_{\eta,\alpha} = \Bigg\{(\sigma,\varphi) : \text{there exists } r \in I \text{ such that} \\ |\{y \in C_r : M_y(\varphi-\sigma) \le \sqrt{\alpha \log N}\}| \ge \eta |C_r|\Bigg\}, \tag{2.13}$$

rather than the $\mathbb{P}$–probability of $E^\sigma_{\eta,\alpha}$. One may already find this estimate, precisely (2.16) below, in [6] (with a slightly different proof) and in [12]: we give the details for completeness. By the markovianity of φ and σ,

$$\mathbf{P}\otimes\mathbb{P}\left(\Omega^+_{\Lambda\cup\Lambda_c,\sigma}\right) = \mathbf{E}\otimes\mathbb{E}\left[\prod_{y\in\Lambda_c} \mathbf{P}\otimes\mathbb{P}\left(\varphi_y-\sigma_y \ge 0 \middle| \mathcal{F}^{\sigma,\varphi}_{B(y)}\right); \Omega^+_\Lambda\right]. \tag{2.14}$$

Observe now that, under $\mathbf{P}\otimes\mathbb{P}(\,\cdot\,|\mathcal{F}^{\sigma,\varphi}_{B(y)})(\psi,\rho)$,

$$\varphi_y - \sigma_y \sim \mathcal{N}\Bigg(\sum_{z\in B(y)} q(z)(\psi_z-\rho_z), 2G_L\Bigg),$$

where G_L is a positive number with the property that $G_L \nearrow G$ as $L \nearrow \infty$ (of course G_ε of §2.1 and G_L are two different quantities). Now let us choose $\alpha < 8G_L$ and observe that on $E_{\eta,\alpha}$

$$\prod_{y\in\Lambda_c} \mathbf{P}\otimes\mathbb{P}\left(\varphi_y-\sigma_y \ge 0 \middle| \mathcal{F}^{\sigma,\varphi}_{B(y)}\right) \\ = \prod_{y\in\Lambda_c}\left(1-\Phi\left(-\frac{M_y(\varphi-\sigma)}{\sqrt{2G_L}}\right)\right) \le \left(1-N^{-\frac{\alpha+\varepsilon'}{4G_L}}\right)^{\eta|C_r|}, \tag{2.15}$$

where r is any element of I and $\varepsilon' \in (0, 8G_L-\alpha)$. Then for sufficiently large N, a suitable constant c' and some $\varepsilon > 0$, we have that

$$\mathbf{P}\otimes\mathbb{P}\left(\Omega^+_{N,\sigma} \cap E_{\eta,\alpha}\right) \le \exp\left(-cN^{d-2+\varepsilon}\right). \tag{2.16}$$

Transferring this annealed estimate to a quenched one is a matter of applying Lemma 2.2: $\mathbf{P}(\,\mathrm{d}\sigma)$–a.s. for N sufficiently large

$$\mathbb{P}\left(\exists r \in I \text{ s.t. } \left|\left\{y \in C_r : M_y(\varphi) \le \sqrt{\alpha \log N} + M_y(\sigma)\right\}\right| \ge \eta |C_r|;\, \Omega^+_{N,\sigma}\right) \le \exp(-cN^{d-2+\varepsilon}/2). \quad (2.17)$$

We use now the fact that $\left\{(\varphi-\sigma)_y - M_y(\varphi-\sigma)\right\}_{y\in\Lambda_c}$ is an IID family of random variables with variance bounded by G: large deviations for such a family of random variables have speed N^d and therefore for our purposes they are negligible. In particular the estimate (2.17) holds also if we replace the event with $\Omega^+_{\Lambda,\sigma} \cap E^\sigma_{\eta,\alpha}$, $E^\sigma_{\eta,\alpha}$ is defined in (2.11), possibly by choosing α slightly smaller and η slightly larger, thus by equation (2.10) and in view of the result that we want to obtain we may restrict to $\Omega^+_{\Lambda,\sigma} \cap (E^\sigma_{\eta,\alpha})^\complement$. On $\Omega^+_{\Lambda,\sigma} \cap (E^\sigma_{\eta,\alpha})^\complement$, for every $r \in I$ there are at least $(1-\eta)|C_r|$ sites $y \in C_r$ such that $\varphi_y > \sqrt{\alpha \log N} + \sigma_y$ and in the remaining (at most $\eta|C_r|$) sites $\varphi_y \ge \sigma_y$ anyway. Therefore for every choice of $f_r \ge 0$, $r \in I$,

$$\sum_{r\in I} f_r \frac{1}{|C_r|} \sum_{y\in C_r} \varphi_y \ge (1-\eta)\sqrt{\alpha \log N} \sum_{r\in I} f_r - \sum_{r\in I} f_r \frac{1}{|C_r|} \left| \sum_{y \in C_r} \sigma_y \right| . \quad (2.18)$$

We observe that by the multidimensional ergodic theorem (see e.g. [11, Appendix 14.A]), $\mathbf{P}(\,\mathrm{d}\sigma)$–a.s., for every $r \in I(\theta)$, we have that $|\sum_{y\in C_r} \sigma_y|/|C_r| \stackrel{N\to\infty}{\longrightarrow} 0$, thus for N sufficiently large

$$\sum_{r\in I} f_r \frac{1}{|C_r|} \sum_{y\in C_r} \varphi_y \ge (1-2\eta)\sqrt{\alpha \log N} \sum_{r\in I} f_r. \quad (2.19)$$

What we have shown is that the (upper and leading) Laplace asymptotics of $\mathbb{P}\left(\Omega^+_{N,\sigma}\right)$ are dominated by the Laplace asymptotics of the probability of the event specified by the condition (2.19). Therefore what is left is a Gaussian computation and the result is obtained by performing, in order, the limit as $N \to \infty$, $\alpha \nearrow 4G_L$, $L \nearrow \infty$, $\eta \searrow 0$ and $\theta \searrow 0$. The optimization over the choices of $\{f_r\}_{r\in I}$ with the use of the second line in (1.3) yields the capacity term. The details of this part may be found in [6], but also in [2] or [12]. □

Remark 2.3. Let us try to put in words the proof we just presented: the initial and, to a certain extent, key step in the proof of the upper bound for (1.5) is to establish that the repulsion event restricted to trajectories φ which stay below the level $\approx \sqrt{4G \log N}$, in a positive (even very small) density of sites, is negligible. It is then sufficient to work with the rest of the trajectories and for these it is possible, via a nice trick, to reduce the estimate to a Gaussian computation [6]. Using the fact that $\varphi - \sigma$ is still an harmonic crystal in the annealed case, one gets immediately the corresponding annealed estimate involving $\varphi - \sigma$, instead of φ alone, and so $\varphi - \sigma$ is forced to be roughly at distance at least $\approx \sqrt{8G \log N}$ on a set of density close to one. The estimate is then passed to the quenched set–up by quenched–annealed comparison and then transfered to the condition that φ has to be at least as large as $\approx \sqrt{8G \log N}$ on a set of density close to one, since by ergodicity σ is of order 1 on a set of density one.

2.3 Path estimates: proof of Proposition 1.4

The proof of the upper bound on the probability essentially contains already the proof of the lower bound on the height. We know in fact that for $\varepsilon > 0$, $\eta > 0$, $\alpha < 8G$ and L sufficiently large we have that

$$\limsup_{N\to\infty} \frac{1}{N^{d-2+\varepsilon}} \log \mathbb{P}\left(E^{\sigma}_{\eta,\alpha} \cap \Omega^+_{N,\sigma}\right) < 0. \tag{2.20}$$

$\mathbf{P}(\,\mathrm{d}\sigma)$–a.s.. Notice that this result, needed for the precise probability asymptotics is beyond what we need: for the path estimate it is sufficient to remark that if $\varphi \in E^{\sigma}_{\eta,\alpha}$ then there are at least $(1-2\eta)|\Lambda_c|$ sites of Λ_c for which $\varphi_y \geq \sqrt{\alpha \log N} + \sigma_y$. The ingredients to conclude at this point are:

- by the probability lower bound we know that $-\log \mathbb{P}\left(\Omega^+_{N,\sigma}\right)$ is of the order of $N^{d-2} \log N$ so that dividing the probability in left–hand side of (2.20) by $\mathbb{P}\left(\Omega^+_{N,\sigma}\right)$ (and thus conditioning with respect to $\Omega^+_{N,\sigma}$) does not change the estimate;

- the fact of having the estimate only for the sublattice Λ_c is not a problem: it suffices to repeat a finite number ($O(L^d)$) of times this argument by shifting the set of centers Λ_c;

- again by ergodicity we know that, $\mathbf{P}(\,\mathrm{d}\sigma)$–a.s.,

$$|\{x \in D_N \,:\, \sigma_x \geq M\}| \,/|D_N| \stackrel{N\to\infty}{\longrightarrow} \mathbf{P}(\sigma_0 \geq M)$$

which of course vanishes as M tends to infinity. That is, σ can be *large* only on a vanishing density subset of D_N.

Putting everything together one directly obtains that for every $\delta > 0$ and $\alpha < 8G$, $\mathbf{P}(\mathrm{d}\sigma)$–a.s.

$$\lim_{N\to\infty} \mathbb{P}\left(|\{x \in D_N : \varphi_x < \sqrt{\alpha \log N}\}| \geq \delta |D_N| \,\middle|\, \Omega^+_{N,\sigma}\right) = 0. \tag{2.21}$$

For what concerns the upper bound on the height, the ingredient missing in the proof that we presented in [2] is Lemma 2.1, the rest requires no change. □

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